A free-electron laser (FEL) may be created by the interaction of an electron bunch and a nearly copropagating laser pulse, where the laser pulse is sheared to extend the interaction length. For a typical application where the angle between the propagation directions of the bunch and pump laser greatly exceeds $1/\gamma$, where $\gamma$ is the relativistic factor, the output radiation is nearly on the beam axis. For this case, we approximate the laser FEL by a magnetostatic undulator that is aligned with the beam axis.

For parameters of a circularly polarized laser-undulator FEL with output radiation with wavelength of 1 nm, we use this magnetostatic undulator to calculate gain using the one-dimensional model, as well as a three-dimensional model that considers the gain degradation due to diffraction and the electron axial velocity spread due to slice emittance and slice energy spread.
1. Introduction

When an electromagnetic plane wave collides head-on with an ultrarelativistic electron bunch, a free-electron laser (FEL) [1] may be obtained which amplifies x-rays propagating downstream whose frequency is approximately $4\gamma^2$ times that of the incident “pump” wave [2–3], where $\gamma$ is the electron bunch’s relativistic factor. To obtain much softer x-rays, one can propagate a pump wave in nearly the same direction as the electron beam. By using a sheared laser pulse [4–5], the electromagnetic field affecting the bunch can be made comparable to a plane wave over an extended interaction length [6–9].

In this technical note, we consider the interaction of an electron beam with kinetic energy of 300 MeV with a laser pulse with wavelength of 750 nm or 1.04 $\mu$m, for production of x-rays with wavelength of 1 nm. For this case, a magnetostatic undulator FEL that is aligned with the electron beam is used to approximate the laser undulator. We calculate gain using the one-dimensional model as well as a three-dimensional model provided by the numerical fits of M. Xie [10].

2. Wavelength and propagation direction of the output radiation

In the one-dimensional (1D) model, FEL gain is predicted for a uniform density of cold electrons interacting with a plane electromagnetic wave. In the laboratory frame, consider a uniform density of cold electrons drifting in the $z$-direction with average velocity $\beta c$, where $c$ is the speed of light, interacting with a uniform pump electromagnetic wave whose angular frequency is $\omega_{\text{pump}}$, which is propagating in the direction $\theta_{\text{pump}}$ w.r.t. the $z$-axis.

In the beam frame where the average electron velocity is zero, the 1D model predicts that for many cycles of interaction, the maximum gain occurs for a wave that is a 180° reflection of the pump wave [11]. Consequently, the maximum FEL gain is predicted to occur for x-rays which, in the laboratory frame, have angular frequency $\omega_{\text{x-ray}}$ and propagation angle $\theta_{\text{x-ray}}$ obeying [12]

$$\omega_{\text{x-ray}} = \gamma^2 \omega_{\text{pump}} \left(1 + \beta^2 - 2\beta \cos \theta_{\text{pump}}\right)$$

and

$$\cos \theta_{\text{x-ray}} = \frac{2\beta - (1 + \beta^2) \cos \theta_{\text{pump}}}{1 + \beta^2 - 2\beta \cos \theta_{\text{pump}}}$$,

where the average longitudinal relativistic factor is defined by $\gamma^* = (1 - \beta^2)^{-1/2}$.

3. Approximation by an axially aligned magnetostatic undulator

In the practical case where the pump propagation direction differs sufficiently from that of the bunch so that $\theta_{\text{pump}} > 1/\gamma$, the maximum gain according to the 1D model occurs at an angle w.r.t. the electron propagation direction that obeys $\theta_{\text{x-ray}} < 1/\gamma$ [12]. When the undulator strength $K$ does not greatly exceed one, $\theta_{\text{x-ray}} < 1/\gamma$ implies that $K\theta_{\text{x-ray}} < 1/\gamma$, in which case the FEL gain for a wave traveling in the axial direction is nearly equal to the maximum gain occurring in the direction $\theta_{\text{x-ray}}$ [13]. If the electron beam radius is so small that a wave growing in the direction $\theta_{\text{x-ray}}$ runs out of gain medium provided by the beam before saturation, nearly identical gain will occur for an axially propagating wave.
For this practical case, we approximate the FEL by a magnetostatic undulator aligned in the direction of beam propagation, whose xray wavelength equals that of the laser undulator [12].

The 1D model approximates a plane-wave interacting with the electrons. To increase the laser’s interaction with a bunch for a given laser pulse energy, a sheared laser pulse may be used so that the bunch experiences an approximately plane-wave interaction for many cycles [6–9]. In principle, a crab cavity can be used to align the bunch so that it is tilted in the direction of maximum gain. Consequently, the gain medium provided by the electrons is aligned in the direction of maximum gain.

If a crab cavity is not used, the gain medium provided by the electron bunch extends furthest in the axial direction, while for the practical case where \( K\theta_{\text{xray}} \ll 1/\gamma \), the gain in the axial direction is nearly the same as that in the direction \( \theta_{\text{xray}} \) [13]. For this practical case, the SASE output is expected to propagate in a nearly axial direction with propagation angle \( \ll 1/\gamma \), with gain nearly equal to that of the axially-aligned magnetostatic undulator. For a seeded laser-undulator FEL, an axially propagating xray is amplified by nearly the same amount as the axially aligned magnetostatic undulator.

4. Three-dimensional modeling

We consider an FEL with output wavelength of \( \lambda_{\text{xray}} = 1 \) nm produced by a 300.511-MeV electron beam interacting with a 400-period circularly polarized laser undulator with \( K = 0.419 \). The laser undulator may be produced by a 750-nm pump laser propagating at an angle of \( \theta_{\text{pump}} = 0.05049 \) rad w.r.t. the electron beam, in which case the 1D FEL model predicts maximum gain at the angle \( \theta_{\text{xray}} = 67 \) μrad. Alternatively, the laser undulator may be produced by a 1.04-μm pump laser propagating at an angle of \( \theta_{\text{pump}} = 0.05946 \) rad, with \( \theta_{\text{xray}} = 57 \) μrad. In both cases, \( K\theta_{\text{xray}} \ll 1/\gamma = 1.7 \) mrad, so we approximate the FEL with an axially aligned magnetostatic undulator.

In Ref. [10], M. Xie gives numerical fits to FEL simulations which provide a 3D model for the degradation of FEL performance from diffraction and the electron axial velocity spread due to slice emittance and slice energy spread. Using the 1D and 3D models for an axially aligned magnetostatic undulator, we evaluate gain for the parameters given in Table 1. We model an axially aligned helical magnetostatic undulator with period of 0.589 mm and length of 0.2356 m.

We consider the ideal case where a cylindrically symmetric electron beam is focused in the center of the magnetostatic undulator with horizontal and vertical beta functions equal to one-half of the undulator length. In this ideal case, the average transverse beam size in the undulator is given by modeling constant horizontal and vertical beta functions that equal 66% of the length of the undulator.

Table 2 shows the results of evaluating the Pierce parameter, exponential gain length, gain saturation length, and saturation power using the 1D and 3D models. Comparing the 1D and 3D models shows that the gain length is nearly doubled by the bunch’s energy spread and emittance. According to the 3D model, there are 6.9 gain lengths within the 0.2356-m undulator, and the FEL gain is \( e^{6.9} \approx 1,000 \). SASE saturation, which requires interaction over 18 gain lengths, is not achieved.

To examine the dependencies upon the parameters in Table 1, parameter scans are shown in Fig. 1. In Fig. 1(a), the curves show the calculated Pierce parameter \( \rho \) vs. slice energy spread, while the dots show the values for the Table 1 parameters. In Fig. 1(b), slice emittance is varied, while the beam current is varied in Fig. 1(c).

Figure 1(d) shows the Pierce parameter for various values of electron beam energy, characterized by the relativistic parameter \( \gamma \), for a fixed slice energy spread of 300.511 keV and normalized emittance of 0.047 mm-mrad. The undulator period was adjusted to obtain output xray wavelength of 1 nm for each
value of beam energy. Although the undulator length was varied, the horizontal and vertical beta functions were maintained at 0.1555 m.

Figure 1(e) shows the Pierce parameter for various values of the undulator parameter $K$. The undulator period was again adjusted to obtain output xray wavelength of 1 nm for each value of $K$. Beta functions of 0.1555 m were modeled.

Figures 2, 3 and 4 show the dependence of gain length, saturation length and saturation power upon the same parameters.

Figures 1–4 show that the FEL performance depends upon the scanned parameters in a smooth fashion. For the case in Table 1, the 1D Pierce parameter of 0.00153 is comparable to the relative slice energy spread of 0.001, so that the FEL gain length increases substantially if the energy spread is increased, as shown in Fig. 2(a). Figure 2(b) shows that increasing the normalized emittance from 0.047 mm-mrad to 0.2 mm-mrad triples the 3D gain length.

Because of the smooth dependencies, a small degradation in one of the FEL parameters can be compensated by an improvement in another parameter. For example, the gain length may be maintained despite a larger electron-bunch emittance if the undulator $K$ parameter is increased by using a larger pump laser. The 3D Model of an axially aligned magnetostatic undulator allows an estimation of the tradeoff between the different FEL parameters.

5. Summary

For a 400-period laser-undulator FEL created by interaction of an electron bunch with a nearly copropagating laser pulse, the FEL gain and saturation have been calculated by approximating the laser undulator with an axially aligned magnetostatic undulator. When a 3D model is used to include the effects of the electron beam emittance and energy spread, xrays with a wavelength of 1 nm are predicted to be amplified by $e^{6.9} = 1,000$ within the 0.2356-m interaction length.

References
### Table 1. Electron beam and undulator parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>Beam energy</td>
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<tr>
<td>Slice energy spread</td>
<td>300.511 keV</td>
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<tr>
<td>Slice emittance (normalized)</td>
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<td>Peak current</td>
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<td>$\beta$-function value in undulator ($\beta_x = \beta_y$)</td>
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<td>Undulator $K$ value</td>
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<td>Output xray wavelength</td>
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### Table 2. FEL performance

<table>
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<tr>
<th>Gain parameter</th>
<th>1D model</th>
<th>3D model</th>
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<tr>
<td>Pierce parameter ($\rho$)</td>
<td>0.00153</td>
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<td>Gain length [mm]</td>
<td>17.7</td>
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<td>Gain saturation length [m]</td>
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<td>Saturation power [GW]</td>
<td>0.368</td>
<td>0.156</td>
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</table>
Figure 1. The Pierce parameter “ρ” is evaluated using the 1D and 3D models. (a) Evaluation for various values of the electron bunch’s slice energy spread, with the other parameters given in Table 1. (b) Evaluation for various values of the electron bunch’s slice emittance, with the other parameters given in Table 1. (c) Evaluation for various values of the electron bunch’s peak beam current, with the other parameters given in Table 1. (d) Evaluation for various values of the electron bunch energy (characterized by relativistic factor γ). An energy spread of 300.511 keV was considered, while the undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1. (e) Evaluation for various values of the undulator K parameter. The undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1.
(a) gain length vs energy spread

(b) gain length vs emittance

(c) gain length vs beam current
Figure 2. The gain length is evaluated using the 1D and 3D models. (a) Evaluation for various values of the electron bunch’s slice energy spread, with the other parameters given in Table 1. (b) Evaluation for various values of the electron bunch’s slice emittance, with the other parameters given in Table 1. (c) Evaluation for various values of the electron bunch’s peak beam current, with the other parameters given in Table 1. (d) Evaluation for various values of the electron bunch energy (characterized by relativistic factor $\gamma$). An energy spread of 300.511 keV was considered, while the undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1. (e) Evaluation for various values of the undulator $K$ parameter. The undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1.
(a) Saturation length vs energy spread

(b) Saturation length vs emittance

(c) Saturation length vs beam current
Figure 3. The gain saturation length is evaluated using the 1D and 3D models. (a) Evaluation for various values of the electron bunch’s slice energy spread, with the other parameters given in Table 1. (b) Evaluation for various values of the electron bunch’s slice emittance, with the other parameters given in Table 1. (c) Evaluation for various values of the electron bunch’s peak beam current, with the other parameters given in Table 1. (d) Evaluation for various values of the electron bunch energy (characterized by relativistic factor $\gamma$). An energy spread of 300.511 keV was considered, while the undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1. (e) Evaluation for various values of the undulator $K$ parameter. The undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1.
Figure 4. The saturation power is evaluated using the 1D and 3D models. (a) Evaluation for various values of the electron bunch’s slice energy spread, with the other parameters given in Table 1. (b) Evaluation for various values of the electron bunch’s slice emittance, with the other parameters given in Table 1. (c) Evaluation for various values of the electron bunch’s peak beam current, with the other parameters given in Table 1. (d) Evaluation for various values of the electron bunch energy (characterized by relativistic factor $\gamma$). An energy spread of 300.511 keV was considered, while the undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1. (e) Evaluation for various values of the undulator $K$ parameter. The undulator period was varied to maintain output xray wavelength of 1 nm. The other parameters are given in Table 1.