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University of Wisconsin-Synchrotron Radiation Center
TECHNICAL NOTE

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Subject: Tune measurement using driven transverse oscillations

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The Aladdin tunes are measured by driving transverse oscillations with a tune excitation electrode and measuring the beam response with a tune pickup electrode. To analyze this process, we consider damping of coherent betatron oscillations from the betatron tune spread and radiation damping. We show that the horizontal tune signal is proportional to the square root of the product of the horizontal betatron functions at the excitation and pickup locations, while the vertical tune signal is proportional to the square root of the product of the vertical betatron functions at the excitation and pickup locations. The signal is independent of the betatron phase advances between the two locations.

Consequently, moving the Aladdin tune excitation electrode from Short Straight Section (SSS) 11 to SSS 12 is predicted to have only a minor effect upon the tune measurement, consistent with observations.
1. Introduction

To analyze the measurement of betatron tunes by using driven resonant transverse oscillations, the tune spread and radiation damping are considered. For a single-peak tune distribution that can be approximated by a Cauchy distribution, the phase-mix damping of coherent oscillations results in an exponential decay, so that both the tune spread and radiation damping may be approximated by linear damping. For a small driven oscillation from a tune excitation electrode, we describe the beam response at the position of a tune pickup electrode.

For resonant excitation, the horizontal tune signal is proportional to the square root of the product of the horizontal betatron functions at the excitation and pickup locations, while the vertical tune signal is proportional to the square root of the product of the vertical betatron functions at the excitation and pickup locations. The signal is independent of the betatron phase advances between the two locations.

We consider the movement of the tune excitation electrode from Short-Straight-Section (SSS) 11 to SSS 12. For the most commonly used magnet lattices (LF15 and Base lattice), the square roots of the horizontal and vertical betatron functions in SSS 11 and SSS 12 differ by less than 30%, so that moving the tune excitation electrode is predicted to modify the tune signals by less than 30%. Therefore, moving the excitation electrode is predicted to be inconsequential, in agreement with observations.

2. Driven oscillations

Consider a driven oscillation in the horizontal or vertical direction for a ring without horizontal-vertical coupling. A kick that bends the orbit through the angle $\theta$ at the longitudinal position $s = 0$ at time $t = 0$ gives excitations at the location $s \in [0, C)$, where $C$ is the ring circumference, arriving at times $t = n T_0 + s / c$, $n = 0, 1, \ldots, \infty$, where $T_0$ is the recirculation time and $c$ is the speed of light.

When the tune spread and radiation damping are neglected, the excitation arriving at time $n T_0 + s / c$ is given by eqs. (2.57) and (2.58) of Ref. [1] as

$$x_n(s) = \sqrt{\beta(0)\beta(s)} \sin[2\pi \nu n + \psi(s)]$$

where $\beta(s)$ is the betatron function, $\nu$ is the betatron tune, and $\psi(s)$ is the betatron phase at the pickup location minus the betatron phase at the excitation location. For radiation-damping rate $\alpha$ (equal to the inverse of the radiation-damping time constant), the damped excitation is

$$x_n(s) = \sqrt{\beta(0)\beta(s)} \sin[2\pi \nu n + \psi(s)] \exp[-\alpha(n T_0 + s / c)]$$

For a Cauchy (also called Lorentzian) distribution of betatron frequencies

$$f(\omega) = (\alpha / \pi) / [((\omega - \omega_0)^2 + \alpha^2)]$$

coherent oscillations decay exponentially from phase-mix damping [2], behaving like the linear damping of oscillations when there is no tune spread [3, 4, 5]. Thus, eq. (2) also approximates the effect of tune spread for a betatron frequency distribution with half width $\alpha$.

For a time-dependent excitation $\theta(t)$, the beam response is

$$x(s, t) = \sum_{n=0}^{\infty} \sqrt{\beta(0)\beta(s)} \theta(t - n T_0 - s / c) \sin[2\pi \nu n + \psi(s)] \exp[-\alpha(n T_0 + s / c)]$$

(3)

For a periodic excitation $\theta(t) = \theta_0 \cos(\omega t + \phi)$,

$$x(s, t) = \sum_{n=0}^{\infty} \sqrt{\beta(0)\beta(s)} \theta_0 \cos[\omega(t - n T_0 - s / c) + \phi] \sin[2\pi \nu n + \psi(s)] \exp[-\alpha(n T_0 + s / c)]$$

(4)

Equation (4) may be written in complex notation as

$$x(s, t) = \theta_0 \sqrt{\beta(0)\beta(s)} e^{i(\omega t + \phi)} \sum_{n=0}^{\infty} e^{-i(\omega - \alpha) s / c n T_0} \sin[2\pi \nu n + \psi(s)],$$

(5)

where the real part of $x(s, t)$ describes the physical oscillation.
Writing the sin term as the difference of two exponentials and using the relation \( \omega_0 T_0 = 2\pi \), where \( \omega_0 \) is the angular revolution frequency, we have
\[
x(s, t) = (\theta_0 / 2i)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i(\omega - \alpha_0 x)/c} \left( e^{i\psi(s)} \sum_{n=0}^{\infty} e^{-(\omega - i\alpha_0 x - v_0 )nT_0} - e^{-i\psi(s)} \sum_{n=0}^{\infty} e^{-(\omega - i\alpha_0 x + v_0 )nT_0} \right)
\]
Using \( \sum_{n=0}^{\infty} e^{inx} = (1 - e^x)^{-1} \), we obtain
\[
x(s, t) = (\theta_0 / 2i)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i(\omega - \alpha_0 x)/c} \left[ \frac{e^{i\psi(s) + i(\omega - v_0 )T_0 / 2}}{\sin[(\omega - v_0 - i\alpha_0 T_0) / 2]} - \frac{e^{-i\psi(s) + i(\omega + v_0 )T_0 / 2}}{\sin[(\omega + v_0 - i\alpha_0 T_0) / 2]} \right]
\]
which may be written as
\[
x(s, t) = (-\theta_0 / 4)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i(\omega - \alpha_0 x)/c} e^{i\alpha T_0 / 2} \left[ \frac{e^{i\psi(s) + i(\pi(\nu - v) - \psi) / 2}}{\sin[\pi(\nu - v) - i\alpha T_0 / 2]} - \frac{e^{-i\psi(s) + i\pi(\nu + v) / 2}}{\sin[\pi(\nu + v) - i\alpha T_0 / 2]} \right]
\]
Defining the tune of the driving frequency \( \nu \) as \( \nu_d = \omega_0 T_0 / 2\pi \), we have
\[
x(s, t) = (-\theta_0 / 4)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i(\omega - \alpha_0 x)/c} e^{i\alpha T_0 / 2} \left[ \frac{e^{i\psi(s) + i\pi(\nu_d - \psi) / 2}}{\sin[\pi(\nu_d - \psi)]} - \frac{e^{-i\psi(s) + i\pi(\nu_d + \psi) / 2}}{\sin[\pi(\nu_d + \psi)]} \right]
\]
Equation (9) is our main result, describing driven oscillations with the effects of damping and/or tune spread of a Cauchy distribution.

When the values of \( \nu_d \pm \nu \) differ from all integers by much more than \( \alpha T_0 / 2\pi \) (which requires that \( \alpha T_0 / 2\pi \ll 1 \)), the damping may be neglected. In this case, eq. (9) becomes
\[
x(s, t) = (-\theta_0 / 4)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i\alpha x / c} \left[ \frac{e^{i\psi(s) + i\pi(\nu_d - \psi) / 2}}{\sin[\pi(\nu_d - \psi)]} - \frac{e^{-i\psi(s) + i\pi(\nu_d + \psi) / 2}}{\sin[\pi(\nu_d + \psi)]} \right]
\]
and one may apply the ac-dipole theory of Refs. [6, 7, 8]. For example, when damping can be neglected and \( \omega = n\omega_0 \) (so that \( \nu_d = n \), we have
\[
x(s, t) = (\theta_0 / 2)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i\alpha x / c} \cos[\psi(s) - \pi \nu] / \sin(\pi \nu)
\]
For the special case \( \omega = \phi = 0 \) (so that \( \nu_d = 0 \)), eq. (11) reproduces the closed-orbit disturbance
\[
x(s, t) = (\theta_0 / 2)\sqrt{\beta(0)\beta(s)} \cos[\psi(s) - \pi \nu] / \sin(\pi \nu)
\]
given by eq. (2.92) of Ref. [1], [where eq. (2.92) is corrected, according to the SLAC-121 Addendum of May 1979, by multiplying by \(-1\)]. In eqs. (11) and (12), the oscillation amplitude depends upon the betatron phase advance \( \psi(s) \) between the excitation location and the pickup location.

When measuring the betatron tunes by resonant excitation, the betatron damping from radiation damping and tune spread cannot be neglected. When \( \nu_d - \nu \approx n \) for some integer \( n \), while \( \nu_d + \nu \) differs from all integers by much more than \( \alpha T_0 / 2\pi \), the first term in eq. (9) dominates so that
\[
x(s, t) \approx (-\theta_0 / 4)\sqrt{\beta(0)\beta(s)}e^{i(\omega_0 + \phi)}e^{-i\alpha x / c} \cos[\psi(s) - \pi \nu] / \sin(\pi \nu)
\]
The amplitude of the oscillation at driving tune \( \nu_d \) is
\[
(\theta_0 / 4)\sqrt{\beta(0)\beta(s)} / |\sin[\pi(\nu_d - \psi) - i\alpha T_0 / 2]|.
\]
The peak oscillation amplitude for resonant excitation with \( \nu_d - \nu = n \) is
\[
\theta_0 \sqrt{\beta(0)\beta(s)} / 2\alpha T_0
\]
For resonant excitation, the peak amplitude at the pickup does not depend upon the phase advance \( \psi(s) \).
The above analysis has considered a single-frequency oscillation. When the tune is measured by sweeping the excitation frequency at rate $d\omega / dt$, the theory of a swept-frequency spectrum analyzer indicates that the measurement bandwidth $\Delta\omega_m$ obeys $(\Delta\omega_m)^2 = d\omega / dt$. For spectral features with width $\Delta\omega > \Delta\omega_m = \sqrt{d\omega / dt}$, the response obeys the above theory of a periodic single-frequency oscillation.

The minimum sweep time for measurement of a feature with bandwidth $\omega_\Delta$ occurs when the frequency is swept over a range $\sim \Delta\omega$ in time $\tau$ with $(\Delta\omega)^2 = \Delta\omega / \tau$, i.e. $\tau \sim 1 / \Delta\omega$. Therefore, measuring the tune peak at $\nu_d = \nu$, where $\Delta\omega = \alpha$, requires a minimum sweep time $\tau$ of $\sim 1 / \alpha$. Measuring the beam response at a driving tune $\nu_d = \nu$, where $\Delta\omega \approx 2 |\sin[\pi(\nu_d - \nu) - i\alpha T_0 / 2]| / T_0$, requires a smaller minimum sweep time of $\sim T_0 / (2 |\sin[\pi(\nu_d - \nu) - i\alpha T_0 / 2]|)$, which is comparable to the minimum measurement time when one ramps the amplitude of a constant-frequency excitation [7].

3. Moving the Aladdin tune excitation electrode

The Aladdin tunes are measured by using excitation and pickup electrodes oriented at $45^\circ$ with respect to the horizontal direction. To place an undulator in SSS 11, the excitation electrode in SSS 11 was moved to SSS 12, where the values of $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$ are within 30% of their values in SSS 11 for the commonly used lattices LF15 and Base lattice. Therefore, moving the tune excitation electrode to SSS 12 is predicted to modify the tune signals by less than 30%, which is inconsequential for ordinary operations. As predicted, moving the excitation electrode did not impact the measurement of betatron tunes.

The Aladdin radiation damping time constants are $\sim 30$ ms, while the measured decoherence of betatron oscillations occurs over $\sim 1000$ turns $= 0.3$ ms [9]. The total betatron damping rate (from radiation damping and tune spread) is $\alpha \sim (0.3 \text{ ms})^{-1}$. The minimum sweep time required for measurement of a single tune peak is therefore $\sim 0.3$ ms. To accurately measure both tune peaks by sweeping over the angular frequency range $\omega_0 (\nu_y - \nu_x) = (2\pi * 3.4 \text{ MHz})(7.234 - 7.139) = 2 \times 10^6 \text{ rad/s}$ requires a sweep time $\tau$ that obeys $d\omega / dt = (2 \times 10^6 \text{ rad/s}) / \tau = (\Delta\omega)^2 \sim (0.3 \text{ ms})^{-2}$, i.e. $\tau \sim 0.2$ s.

For the horizontal tune measurement with $\beta_x(0) \sim \beta_x(s) \sim 4$ m, $\alpha \sim (0.3 \text{ ms})^{-1}$ and $T_0 = 2.96 \times 10^{-7}$ s, eq. (15) gives a peak oscillation amplitude (in meters) of $2000 \theta_0$, where $\theta_0$ is the peak horizontal deflection from the excitation electrode. For the vertical tune measurement with $\beta_x(0) \sim \beta_x(s) \sim 0.7$ m, the peak oscillation amplitude is $350 \theta_0$. Therefore, the peak horizontal and vertical oscillation amplitudes for a periodic deflection of 1 $\mu$rad are $\sim 2000 \mu$m and $\sim 350 \mu$m.